the different vibrational levels of the excited molecule are treated separately. Serious computer time and core storage problems occur, especially when multiquantum deactivation[4] is included. This type of deactivation process replaces the one-step-at-a-time reaction with a larger number that increases linearly with v, where v denotes the vibrational level of the excited molecule. In this note, a technique is introduced which is based on reasonable assumptions, and which contracts these reactions into a smaller number, thereby making their inclusion practical for computer modeling.

Analysis

For convenience, consider a model that treats the first $m+1$ vibrational levels of HF(v). There are then $m(m+1)/2$ multiquantum reactions with deactivator M:

$$\mathrm{HF}(v)+M \underset{k^{v,v'}}{\overset{k_{v,v'}}{\rightleftharpoons}} \mathrm{HF}(v')+M$$

where $1 \le v \le m$, $0 \le v' \le v-1$. Assume that the backward rate coefficient $k^{v,v'}$ is zero, since the exponential factor in the equilibrium expression

$$k^{v,v'} = k_{v,v'} \exp(-\Delta E/RT)$$

is negligibly small for temperatures T of interest. The kinetic rate equations for the concentrations HF(v) then can be written as

$$\frac{d\mathrm{HF}(0)}{dt} = M \sum_{v'=1}^{m} k_{v',0}\mathrm{HF}(v')$$

$$\vdots$$

$$\frac{d\mathrm{HF}(v)}{dt} = M \sum_{v'=v+1}^{m} k_{v',v}\mathrm{HF}(v') - M\cdot\mathrm{HF}(v) \sum_{v'=0}^{v-1} k_{v,v'}$$

$$\vdots$$

$$\frac{d\mathrm{HF}(m)}{dt} = -M\cdot\mathrm{HF}(m) \sum_{v'=0}^{m-1} k_{m,v'} \qquad (1)$$

Next assume that all transitions from a given level v are equally probable, i.e., $k_{v,v'} = k_v$. Additionally, the assumption is made that $a_{v,v'} \equiv (k_{v'}/k_v)$, $v \le v' \le m$, is independent of T, and thus is constant. Both assumptions are in good accord with the multiquantum deactivation rates calculated by Wilkins.[4] The rate coefficients and the two summations in Eq. (1) become

$$k_{v',v} = k_{v'} = a_{1,v'}k_1 \qquad k_{v,v'} = k_v = a_{1,v}k_1$$

$$\sum_{v'=0}^{v-1} k_{v,v'} = va_{1,v}k_1 \qquad \sum_{v'=v+1}^{m} k_{v',v}\mathrm{HF}(v') = k_1 M_v$$

$$M_v \equiv \sum_{v'=v+1}^{m} a_{1,v'}\mathrm{HF}(v') \qquad v=0,\ldots,m-1 \qquad (2)$$

The rate equations now have the form

$$d\mathrm{HF}(0)/dt = k_1 M M_0$$

$$\vdots$$

$$d\mathrm{HF}(v)/dt = k_1 M M_v - (va_{1,v}k_1)M\cdot\mathrm{HF}(v)$$

$$\vdots$$

$$d\mathrm{HF}(m)/dt = -(ma_{1,m}k_1)M\cdot\mathrm{HF}(m)$$

These equations can be constructed directly by nonequilibrium computer codes from the $m+1$ "reactions"

$$\mathrm{HF}(v)+M \underset{k'_{-v}}{\overset{k'_v}{\rightleftharpoons}} M_v+M \qquad v=0,\ldots,m$$

where $k'_v = va_{1,v}k_1$, $k'_{-v} = k_1$, and M_v is given by Eq. (2).

The number of reactions for modeling the multiquantum process thus has been reduced from $m(m+1)/2$ to $m+1$. For HF chain reaction modeling, where m is at least 7, the reduction is from 28 to 8 reactions per process. Furthermore, the number of deactivation processes thought to be multiquantum is steadily increasing. Note that both k'_v and k'_{-v} have the usual Arrhenius temperature-dependent form. Both coefficients are required, and their ratio is not an equilibrium constant. Since the $a_{1,v}$ are constant, the M_v, Eq. (2), have a simple form commonly used in computer codes for collisional deactivators. The new "reactions" do not conserve atoms in the usual fashion. Nevertheless, this formulation is useful, since few, if any, computer programming changes are required for its implementation.

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Simple Waves and the Transonic Similarity Parameter

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Introduction

THE transonic small perturbation equation for the perturbed potential (in two dimensions) has the form

$$\varphi_{yy} = \beta^2\varphi_{xx} + \Gamma(M)\varphi_x\varphi_{xx} \qquad (1)$$

where $\beta^2 = M^2 - 1$, and the potential φ has been normalized with respect to the upstream velocity. In the original derivation of Eq. (1) by von Karman,[1] $\Gamma = \gamma+1$, and in the course of time various other suggestions for $\Gamma(M)$ have been made. Spreiter[2] pointed out that this coefficient is ambiguous [to $0(\beta^2)$] and went on to suggest, $\Gamma_s = (\gamma+1)M^2$, which he demonstrated to be effective in transonic scaling. Hayes,[3] also noting this ambiguity, showed three such naturally occurring coefficients, and particularly noteworthy for our discussion is $\Gamma_H = (\gamma+1)M^4$, which appeared in an earlier paper, Hayes.[4] More recently Murman and Krupp[5] (see also

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Murman[6]) on the basis of an empirical investigation, propose the use of $\Gamma_{MK}=(\gamma+1)M^{3/2}$ which they show to be most effective in transonic similarity scaling.

Transonic scaling naturally takes place through the similarity parameter

$$K=\frac{\beta^2}{[\epsilon\Gamma(M)]^{3/2}} \tag{2}$$

ϵ being the smallness ratio in the problem, and in the following we investigate the ambiguity in Γ by analyzing the simple wave structure of the potential equation in the neighborhood of near sonic flow. In particular it is shown that within this framework an unambiguous choice of $\Gamma(M)$ can be made. But that in order to do so one must appeal to higher order terms of the full potential equation.

Simple Wave Analysis

We consider two-dimensional flow past a body located at $y=\epsilon f(x)$, where ϵ is the thickness ratio and the body is assumed to have unit length. The boundary condition $v=\epsilon f'(x)(1+u)$ is applied at the body and to second order the governing equations are

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \tag{3}$$

$$\frac{\partial v}{\partial y}=\beta^2\frac{\partial u}{\partial x}+M^2(\gamma+1)u\frac{\partial u}{\partial x}+\left\{(\gamma-1)M^2u\frac{\partial v}{\partial y}+M^2 2v\frac{\partial u}{\partial y}+M^2\frac{\gamma+1}{2}u^2\frac{\partial u}{\partial x}\right\} \tag{4}$$

This form, under a different normalization, was given by Hayes[3] (u and v in Eqs. (3) and (4) are normalized with respect to the upstream velocity). In order to obtain Eq. (4) from the full potential equation one uses $u=0(\epsilon^{2/3}), v=0(\epsilon)$ and $\beta^2/\epsilon^{2/3}=0(1)$, all of which are well-known transonic approximations (see Cole[7]).

It is clear that the curly bracket term of Eq. (4) is higher order and if we back substitute for the y-derivatives appearing in it, Eq. (4) becomes

$$\frac{\partial v}{\partial y}=\beta^2\frac{\partial u}{\partial x}+2M^2\alpha u\frac{\partial u}{\partial x}+2M^2\left\{\alpha_0 a u^2\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right\} \tag{5}$$

where

$$\alpha=1+\frac{\gamma-1}{2}M^2,\quad \alpha_0=\alpha(M=1)=\frac{\gamma+1}{2},\quad a=\tfrac{1}{2}+(\gamma-1)M^2$$

The coefficient of u^2u_x in Eq. (4) is ambiguous to $0(\beta^2)$ and is merely retained in the form which is produced by the iteration.

Simple wave solutions to Eqs. (3) and (5) can be directly investigated by assuming $v=v(u)$, which yields

$$u_y=v'(u)u_x \tag{6a}$$

$$v'u_x=\beta^2u_x+2M^2\alpha u+2M^2\{\alpha_0 a u^2+vv'\}u_x \tag{6b}$$

and which in order to have a nontrivial solution must satisfy

$$\left(\frac{dv}{du}\right)^2=\beta^2+2M^2\alpha u+2M^2\left\{\alpha_0 a u^2+v\frac{dv}{du}\right\} \tag{7}$$

The left-hand side of Eq. (7) is nonnegative and hence the analysis breaks down if its right-hand side becomes negative (i.e., a simple wave no longer exists). Therefore, breakdown occurs if $v'=0$ equivalently if $u=u_0$ is such that

$$\beta^2+2M^2(\alpha u_0+\alpha_0 a u_0^2)=0 \tag{8}$$

This is a criterion for solution only for small values of β^2; elsewhere neglected terms become the same order as $\alpha_0 a u^2$. Solving Eq. (8) for $\beta^2\approx 0$ we obtain

$$u_0=-\frac{\beta^2}{\gamma+1}+\frac{(2\gamma+1)\beta^4}{2(\gamma+1)^2}+0(\beta^6) \tag{9}$$

Equation (9) cannot be carried beyond $0(\beta^4)$ since already neglected terms then enter.

Defining

$$\bar{\alpha}=\frac{(2\gamma+1)M^2+1}{2} \tag{10}$$

we can rewrite Eq. (7) as

$$\left(\frac{dv}{du}\right)^2=\beta^2+\bar{\alpha}u+2M^2\left\{\alpha_0 a u^2+\frac{a\beta^2}{2M^2}u+v\frac{dv}{du}\right\} \tag{11}$$

With this choice $\beta^2+\bar{\alpha}u=0$ has the solution (9). Otherwise said, the curly bracket of Eq. (11) and $\beta^2+\bar{\alpha}u$ vanish up to $0(\beta^6)$ as v' vanishes. Alternately, one may solve Eq. (7) by perturbation, say by writing $u=\epsilon^{2/3}\mu, v=\epsilon v_1+\epsilon^{5/3}v_2+\ldots$, $\beta^2=0(\epsilon^{2/3})$. In this case the terms in the curly brackets of Eq. (7) are of higher order. One then finds a divergence in v_2 – which is eliminated if the terms are rearranged as in Eq. (11).

Therefore within the framework of simple wave theory and unambiguous choice of the coefficient of u up to an including $0(\beta^2)$, has been obtained. This in turn implies that the appropriate form of the small perturbation equation is

$$-\beta^2\varphi_{xx}+\varphi_{yy}=\frac{(2\gamma+1)M^2+1}{2}\varphi_x\varphi_{xx} \tag{12}$$

This gives rise to the following form of the transonic similarity parameter

$$K_0=\beta^2/[\epsilon\frac{(2\gamma+1)M^2+1}{2}]^{2/3} \tag{13}$$

Figure 1 contains a comparison between Eq. (13) and the similarity parameter of Spreiter[2]

$$K_s=\beta^2/[\epsilon(\gamma+1)M^2]^{2/3} \tag{14}$$

and that of Murman and Krupp[5]

$$K_{MK}=\beta^2/M[\epsilon(\gamma+1)]^{2/3} \tag{15}$$

Over a significant transonic range ($0.8\le M\le 1.4$) Eqs. (13) and (14) show less than 8% departure and the agreement of

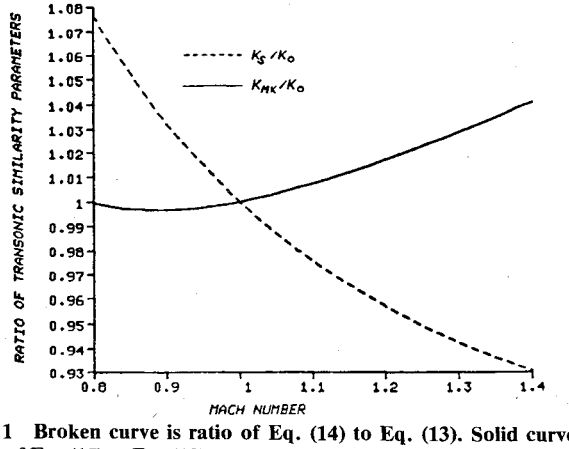

Fig. 1 Broken curve is ratio of Eq. (14) to Eq. (13). Solid curve is ratio of Eq. (15) to Eq. (13).

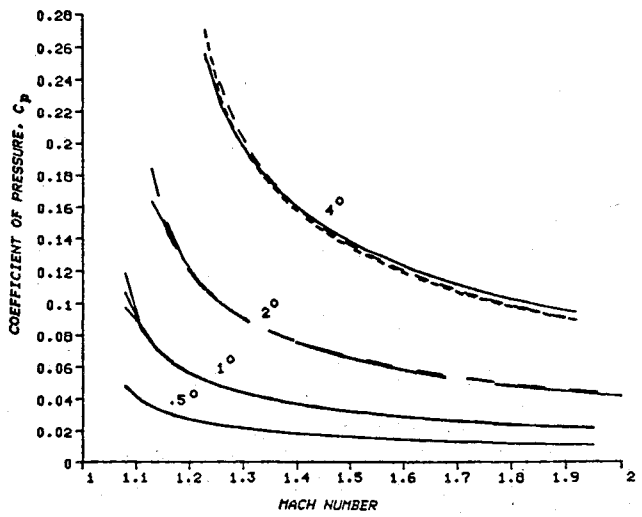

Fig. 2 Solid curve gives pressure coefficient as compiled from oblique shock theory. Short dashed curve as given by Eq. (17); long dashed curve as given by Eq. (18).

Eqs. (13) and (15) is significantly better. In the latter case the departure for $0.8 \leq M \leq 1.1$ is only a fraction of 1%. In view of the excellent agreement with transonic data found with Eq. (15) (Murman[6]), similar agreement should also be found using Eq. (13).

Returning to Eq. (11) and neglecting the higher order terms contained in the curly brackets, we may integrate to obtain

$$v = \frac{2}{3\alpha}\left[\beta^3 - (\beta^2 + \alpha u)^{3/2}\right] \tag{16}$$

The negative branch of the square root was taken, corresponding to upper half plane flow.

An immediate application of Eq. (16) is the computation of the pressure coefficient at a body. The pressure coefficient has the form $C_p = -2u$ in the present approximation; thus solving Eq. (16) and substituting we obtain

$$C_p = \frac{2}{\alpha}\left[\beta^2 - (\beta^3 - \frac{3}{2}\alpha\theta)^{2/3}\right] \tag{17}$$

where we have introduced the linearized boundary condition $v = \theta[\theta = \tan^{-1}(f'(x)]; f(x)$ the body shape).

Spreiter and Alksne[8] suggest that the pressure coefficient for transonic flow in the low supersonic limit is

$$C_{SA} = \frac{2}{(\gamma+1)M^2}\left\{\beta^2 - [\beta^3 - \frac{3}{2}(\gamma+1)M^2\theta]^{2/3}\right\} \tag{18}$$

It is convenient to discuss these results in terms of flow past a wedge, in which case an exact result for the pressure coefficient is known from oblique shock theory. Figure 2 contains plots of the pressure coefficient at the four small angles of 0.5°, 1°, 2°, and 4°, and in each case exact oblique shock theory as well as Eqs. (17) and (18) are shown. In each case Eq. (17) lies closer than Eq. (18) to oblique shock theory, however the numerical differences are quite small.

One further point to note in Fig. 2 is that in the case of a 4° wedge both Eqs. (17) and (18) begin to depart seriously from the exact theory at higher Mach numbers. This might have been anticipated since neither Eq. (12) nor Eq. (1) with $\Gamma = (\gamma+1)M^2$ are appropriate at supersonic speeds. Under this limit Hayes' equation, i.e., Eq. (1) with $\Gamma = (\gamma+1)M^4$ is more appropriate. In fact performing the same analysis on the Hayes equation we obtain

$$C_p = \frac{2}{(\gamma+1)M^4}\left\{\beta^2 - [\beta^3 - \frac{3}{2}(\gamma+1)M^4\theta]^{2/3}\right\}$$

which under the supersonic limit $K \to \infty$, or equivalently θ small, becomes

$$C_p \sim \frac{2}{\beta}\theta + \frac{\gamma M^4 + (M^2-2)^2}{2\beta^4}\theta^2$$

This will be recognized as the Busemann second-order expansion (Busemann[9], Lighthill[10]). Therefore, said in other terms the discrepancy of Eqs. (17) and (18) with exact theory at higher Mach numbers is due to the fact that both of these are only valid to first order in that limit.

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Laminar Compressible Boundary Layers with Vectored Mass Transfer

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Introduction

THE study of the effects of surface mass transfer involving both normal and tangential components (vectored suction or injection) on bodies is useful in understanding several important boundary-layer problems such as transpiration cooling of rocket engines and turbine blades, mixing processes in pollution problems, flow on reentry bodies, and many other similar problems.[1] Inger and Swean,[1] and Scala and Sutton[2] have studied the effects of vectored suction or injection on laminar compressible boundary-layer flow of a gas with constant properties (i.e., $\rho\mu$ = constant, $Pr = 1$ and 0.7, where ρ, μ, and Pr are the density, viscosity, and Prandtl number, respectively) at constant pressure, and at an axisymmetric stagnation point, respectively.

In the present study, we have extended the previous analysis of Ref. 2 to include the effects of variable gas properties. In particular, we have studied the effects of vectored suction or

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